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Zero stiffness tensegrity structures

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Abstract

Tension members with a zero rest length allow the construction of tensegrity structures that are in equilibrium along a continuous path of configurations, and thus exhibit mechanism-like properties; equivalently, they have zero stiffness. The zero-stiffness modes are not internal mechanisms, as they involve first-order changes in member length, but are a direct result of the use of the special tension members. These modes correspond to an infinitesimal affine transformation of the structure that preserves the length of conventional members, they hold over finite displacements and are present if and only if the directional vectors of those members lie on a projective conic. This geometric interpretation provides several interesting observations regarding zero stiffness tensegrity structures.

Keywords: Zero stiffness; Tensegrity structures; Tensegrity mechanisms; Static balancing; Affine transformations; Projective conic

1. Introduction

This paper will describe and analyse a new and special class of ‘tensegrity’ structures that straddle the border between mechanisms and structures: although member lengths and orientations change, the structures can be deformed over large displacements whilst continuously remaining in equilibrium. In other words, they remain neutrally stable, require no external work to deform, and hence have zero stiffness. Although zero stiffness is uncommon in the theory of stability, several examples exist. Tarnai (2003) describes two systems that display zero stiffness, respectively, related to bifurcation of equilibrium paths, and to snap-through type loss of stability of unloaded structures in a state of self-stress. These structures require specific external loads or states of self-stress to exhibit zero stiffness. The key to the structures discussed in this paper, however, is the use of tension members that, in their working range, appear to have a zero rest length—their tension is proportional to their length. Such members are not merely a mathematical abstraction; it is for instance possible to wind a close-coiled spring with initial tension that ensures, when the spring is extended, that the exerted force is proportional to the length.

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The utility of zero-free-length springs was initially exploited in the design of the classic ‘Anglepoise’ lamp (French and Widden, 2000), but is more generally applied in the field of static balancing (Herder, 2001)—see Fig. 1. To those unfamiliar with static balancing, it may be surprising that systems such as those shown in Fig. 1(b) and (c) are indeed in equilibrium for any orientation of the rigid bar. However, simple calculations, such as those shown in French and Widden (2000) will show that, if zero-free-length springs with an appropriate stiffness are used, this is indeed the case. Statically balanced systems are in equilibrium in every configuration in their workspace; every configuration has the same potential energy, and the system hence has zero stiffness. As the only forces required to move a statically balanced system are those to overcome friction, and to accelerate and decelerate the system, statically balanced systems are used for energy-efficient design in, for instance, robotics and medical settings. Herder (2001) discovered some basic examples of statically balanced tensegrities, which formed the inspiration for the current research.

‘Tensegrity’ is a term that is not consistently defined in literature, see Motro (1992) for a discussion. Here, we take it to mean free-standing prestressed pin-jointed structures, which are in general both statically and kinematically indeterminate. The state of self-stress ensures that each member carries a non-zero, purely tensile or compressive load, under absence of external loads and constraints. Previously, the analysis of tensegrity structures, either by a structural mechanics approach (e.g., Pellegrino and Calladine, 1986) or a mathematical rigidity theory approach (e.g., Connelly and Whiteley, 1996), has been concerned with whether or not a structure is stable. We shall only consider structures that, were they constructed with conventional tension and compression members, would be prestress stable (i.e., have a positive-definite tangent stiffness matrix, modulo rigid-body motions). The novel feature of this paper is that we then replace some or all of the tension members with zero-free-length springs, in search of zero-stiffness modes.

The zero-stiffness tensegrities described in this paper walk a fine line between structures and mechanisms. Here, we shall refer to them as *tensegrity structures*, as we will be using the tools of structural engineering and not mechanism theory. For other purposes, the term *tensegrity mechanisms* might be more applicable. Practical applications of this new class of structure will most likely also take place on the borderline of structures and mechanisms, such as, for example, deployable structures which are in equilibrium throughout deployment.

There are clear hints to the direction taken in this paper in the affine transformations considered by Connelly and Terrell (1995) or the ‘tensegrity similarity transformation’ considered by Masic et al. (2005). Unlike in those papers, here the affine transformations are translated from a mathematical abstraction into a real physical response of structures that can be constructed.

The paper is laid out as follows. Section 2 recapitulates the equilibrium and stiffness analysis of prestressed structures. In particular it describes the consequences of using zero-free-length springs by means of a recent formulation of the tangent stiffness matrix. Section 3 introduces affine transformations and shows that affine modes which preserve the length of the conventional members are statically balanced zero-stiffness modes valid over finite displacements. The link between the projective conic and the presence of length-preserving affine transformations is discussed in Section 4. In Section 5, an example analysis of a classic tensegrity structure fitted with zero-free-length springs is used to illustrate the theory.

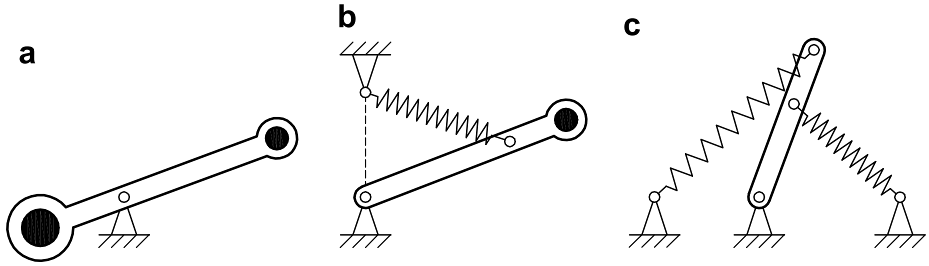


Fig. 1. Static balancing: the three structures shown are in equilibrium for any position of the rigid bar, as long as in (a) the masses (black circles) are in inverse proportion to the distance from the pivot and in (b) and (c) the springs are zero-free-length springs with appropriately chosen stiffness.

2. Equilibrium and stiffness of prestressed structures

This section lays the groundwork for the coming sections, by first briefly recapitulating the tensegrity form-finding method from rigidity theory, followed by a description of the tangent stiffness matrix that clearly shows the effects of using zero-free-length springs. The section is concluded by a discussion of zero-stiffness modes in conventional tensegrity structures.

2.1. Equilibrium position

This paper is primarily concerned with the stiffness of a tensegrity structure in a known configuration, and not with *form finding*, i.e., finding an initial equilibrium configuration (Tibert and Pellegrino, 2003). Nevertheless, a brief description of form finding will be given; there are interesting and useful parallels between the stiffness of a prestressed structure and the *energy method* of rigidity theory (or, equivalently, the engineering *force density method*) used in form finding.

The energy method in rigidity theory considers a stress state ω to be a state of self-stress if the internal forces at every node sum to zero, i.e., the following equilibrium condition holds at each node i

$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = \mathbf{0} \tag{1}$$

where $\mathbf{p}_i$ are the coordinates for node i, and ω_{ij} is the tension in the member connecting nodes i and j, divided by the length of the member; ω_{ij} is referred to as a *stress* in rigidity theory, but is known in engineering as a *force density* or *tension coefficient*. If all the nodal coordinates are written together as a column vector $\mathbf{p}$, $\mathbf{p}^{\mathrm{T}} = [\mathbf{p}_1^{\mathrm{T}}, \mathbf{p}_2^{\mathrm{T}}, \ldots, \mathbf{p}_n^{\mathrm{T}}]$, the equilibrium equations at each node can be combined to obtain the matrix equation

$$\tilde{\mathbf{\Omega}}\mathbf{p} = \mathbf{0} \tag{2}$$

where $\tilde{\mathbf{\Omega}}$ is the *stress matrix* for the entire structure. In fact, because Eq. (1) consists of the same coefficients for each of d dimensions, the stress matrix can be written as the Kronecker product of a *reduced stress matrix* $\mathbf{\Omega}$ and a d-dimensional identity matrix $\mathbf{I}^d$

$$\tilde{\mathbf{\Omega}} = \mathbf{\Omega} \otimes \mathbf{I}^d \tag{3}$$

The coefficients of the reduced stress matrix are then given, from Eq. (1), as

$$\Omega_{ij} = \begin{cases} -\omega_{ij} = -\omega_{ji} & \text{if } i \neq j, \text{ and } \{i,j\} \text{ a member} \\ \sum_{k\neq i}\omega_{ik} & \text{if } i = j \\ 0 & \text{if there is no connection between } i \text{ and } j \end{cases} \tag{4}$$

Although the stress matrix is here defined entirely by *equilibrium* of the structure, we shall see the same matrix recurring in the *stiffness* equations in Section 2.2. This dual role of the stress matrix allows the combination and application of insights from rigidity theory—where the stress matrix has been the object of study—to engineering stiffness analysis.

Form-finding methods require the symmetric matrix $\mathbf{\Omega}$ to have a nullity $\mathcal{N} \geqslant d+1$, and thus for $\tilde{\mathbf{\Omega}}$ a nullity $\mathcal{N} \geqslant d(d+1)$.[1] If the nullity requirement is not met, the only possible configurations of the structure will be in a subspace of a lower dimension. For example, form finding in 3 dimensions would only be able to produce planar equilibrium configurations (Tibert and Pellegrino, 2003). The significance of this requirement will be further elucidated in Section 3, when affine transformations are introduced. If $\tilde{\mathbf{\Omega}}$ has a nullity equal to $d(d+1)$, we shall describe it as being of *maximal rank*.

2.2. Tangent stiffness matrix

Stability analysis considers small changes from an equilibrium position. For a prestressed structure account must be taken not only of the deformation of the elements and the consequent changes in internal tension, but

[1] The nullity of a square matrix is equal to its dimension minus its rank.

also of the effects of the changing geometry on the orientation of already stressed elements. Consider an infinitesimal displacement $\mathbf{d}$, and force perturbation $\mathbf{f}$, where $\mathbf{d}^{\mathrm{T}} = [\mathbf{d}_1^{\mathrm{T}}, \mathbf{d}_2^{\mathrm{T}}, \ldots, \mathbf{d}_n^{\mathrm{T}}]$, $\mathbf{f}^{\mathrm{T}} = [\mathbf{f}_1^{\mathrm{T}}, \mathbf{f}_2^{\mathrm{T}}, \ldots, \mathbf{f}_n^{\mathrm{T}}]$, and $\mathbf{d}_i$, $\mathbf{f}_i$ are the displacement and force perturbation at node i. The column vectors $\mathbf{d}$ and $\mathbf{f}$ are related by the *tangent stiffness matrix* $\mathbf{K}_t$,

$$\mathbf{K}_t\mathbf{d} = \mathbf{f} \tag{5}$$

The tangent stiffness matrix is well-known in structural analysis, and many different formulations for it exist (e.g., Murakami, 2001; Masic et al., 2005). Different formulations with identical underlying assumptions will produce identical numerical results, but may provide a different *understanding* of the stiffness. The formulation used in this paper is derived by Guest (2006); it is written as:

$$\begin{aligned} \mathbf{K}_t &= \hat{\mathbf{K}} + \tilde{\mathbf{\Omega}} \\ &= \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}} + \tilde{\mathbf{\Omega}} \end{aligned} \tag{6}$$

where $\tilde{\mathbf{\Omega}}$ is the stress matrix described earlier and $\hat{\mathbf{K}}$ is the *modified material stiffness matrix*. The modified material stiffness matrix is written in terms of $\mathbf{A}$, the *equilibrium matrix* for the structure, and $\hat{\mathbf{G}}$, a diagonal matrix whose entries consist of the *modified axial stiffness* for each of the members. The modified axial stiffness $\hat{g}$ is defined as

$$\hat{g} = g - \omega \tag{7}$$

where g is the conventional axial stiffness and ω the tension coefficient. For conventional members, $\hat{g}$ will be little different from g. It will certainly always be positive, and hence the matrix $\hat{\mathbf{G}}$ will always be positive definite. However, for a zero-free-length spring, because the tension t is proportional to the length, $t = gl$, the tension coefficient is equal to the axial stiffness, $\omega = t/l = g$, and the modified axial stiffness $\hat{g} = g - \omega = 0$. Thus structures constructed with zero-free-length springs will have zeros along the diagonal of $\hat{\mathbf{G}}$ corresponding to these members, and $\hat{\mathbf{G}}$ will now only be positive *semi*-definite.

Normally, a zero axial stiffness would be equivalent to the removal of that member (Deng and Kwan, 2005). This is not the case for the zero modified axial stiffness of zero-free-length springs, because the contribution of the member is still present in the stress matrix $\tilde{\mathbf{\Omega}}$. This leads to the observation that for zero-free-length springs the geometry (i.e., the equilibrium matrix $\mathbf{A}$) is irrelevant and only the tension coefficient and member connectivity (i.e., the stress matrix $\tilde{\mathbf{\Omega}}$) define their reaction to displacements.

2.3. Zero-stiffness modes and internal mechanisms

The main interest of this paper lies in displacements that have a zero stiffness; in other words, displacements that are in the *kernel*, or *nullspace*, of the tangent stiffness matrix. A zero tangent stiffness for some deformation $\mathbf{d}$ requires, from Eq. (6), either that $\hat{\mathbf{K}}\mathbf{d} = -\tilde{\mathbf{\Omega}}\mathbf{d}$, or that both $\hat{\mathbf{K}}\mathbf{d}$ and $\tilde{\mathbf{\Omega}}\mathbf{d}$ are zero. We will concentrate on the second case, i.e., $\mathbf{d}$ lies in the nullspace of both $\hat{\mathbf{K}}$ and $\tilde{\mathbf{\Omega}}$, but will briefly discuss the other possibility in Section 3.5.

For a conventional structure, as $\hat{\mathbf{G}}$ is positive definite, the nullspace of $\hat{\mathbf{K}} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}$ is equal to the nullspace of $\mathbf{A}^{\mathrm{T}}$, and hence $\mathbf{A}^{\mathrm{T}}\mathbf{d} = \mathbf{0}$. The matrix $\mathbf{C} = \mathbf{A}^{\mathrm{T}}$ is the *compatibility matrix* (closely related to the *rigidity matrix* in rigidity theory) of the structure, and the extension of members $\mathbf{e}$ is given by $\mathbf{Cd} = \mathbf{e}$; i.e., $\mathbf{e} = \mathbf{0}$ for a zero-stiffness mode. Thus, for a conventional structure a zero tangent stiffness requires the deformation to be an *internal mechanism*: a deformation that to first order causes no member elongation. In addition, $\tilde{\mathbf{\Omega}}\mathbf{d}$ must be zero, which implies that the mechanism is not stabilized by the self-stress in the structure. One obvious mode is that rigid-body displacements of the entire structure will have no stiffness. However, in general there may also be other non-stiffened (higher-order) infinitesimal, or even finite, internal mechanisms present (see e.g., Pellegrino and Calladine, 1986; Kangwai and Guest, 1999). Infinitesimal mechanisms may eventually stiffen due to the higher-order elongations of members, but finite internal mechanisms have no stiffness over a finite path. If a structure is prestress stable, all displacements have a positive stiffness. This means that, modulo rigid-body motions, all eigenvalues of the tangent stiffness matrix are positive and the matrix is thus positive definite.

Some of the above observations change when a structure includes zero-free-length springs, which have modified axial stiffness $\hat{g} = 0$. A key observation is that the nullspace of $\hat{\mathbf{K}} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}$ is no longer the same as the nullspace of $\mathbf{A}^{\mathrm{T}}$, as $\hat{\mathbf{G}}$ is now only positive semi-definite. The increased nullity of the modified material stiffness matrix $\hat{\mathbf{K}}$ is of great importance to this study, as it will prove to be key to finding the desired zero-stiffness modes (see Section 3). Note, however, by contrast, that the form of the stress matrix $\tilde{\mathbf{\Omega}}$ is unchanged when zero-free-length springs are introduced to the structure.

We introduce the term '*statically balanced zero-stiffness mode*' to distinguish between zero-stiffness modes found in conventional tensegrity structures, such as internal mechanisms and rigid-body motions, and zero-stiffness modes introduced by the presence of zero-free-length springs. In contrast with internal mechanisms, these latter modes involve first-order changes in member length, and thus energy exchange among the members.

3. Affine transformations and zero-stiffness modes

This section first introduces the concept of affine transformations, and infinitesimal affine modes, before showing that affine modes that preserve the length of 'conventional' members are statically balanced zero-stiffness modes that are valid over finite displacements. It shall further be argued that for prestress stable tensegrity structures with a positive semi-definite stress matrix of maximal rank, these are the only possible zero-stiffness modes.

3.1. Affine transformations

As described in Section 2.1, the equilibrium position of a freestanding tensegrity structure for a given state of self-stress is given by $\tilde{\mathbf{\Omega}}\mathbf{p} = \mathbf{0}$. Under an *affine transformation* of the nodal coordinates $\mathbf{p}$ this condition still holds (Connelly and Whiteley, 1996; Masic et al., 2005), and hence the new geometry is also in equilibrium for the same set of tension coefficients.

Affine transformations are linear transformations of coordinates (of the whole affine plane onto itself) preserving collinearity. Thus, an affine transformation transforms parallel lines into parallel lines and preserves ratios of distances along parallel lines, as well as intermediacy (Coxeter, 1989, pp. 202). Consider nodal coordinates $\bar{\mathbf{p}}_i$ as an affine transformation of the original coordinates $\mathbf{p}_i$. This transformation can be generally described by

$$\bar{\mathbf{p}}_i = \mathbf{U}\mathbf{p}_i + \mathbf{w}$$

where in d-dimensional space $\mathbf{U}$ is an invertible d-by-d matrix, and $\mathbf{w}$ is a d-component column vector. This provides a total of $d(d+1)$ independent affine transformations. Affine transformations are well-known to engineers, but under a different guise. Suppose we split the matrix $\mathbf{U}$ into an orthogonal component $\mathbf{U}_S$, and a component $\mathbf{U}_Q$, such that $\mathbf{U} = \mathbf{U}_S + \mathbf{U}_Q$. Then half of the $d(d+1)$ affine transformations is constituted by $\mathbf{w}$ and $\mathbf{U}_S$, and these are rigid-body motions (e.g., 6 rigid-body motions in 3-dimensional space). The other half, formed by $\mathbf{U}_Q$, is equivalent to the basic strains found in continuum mechanics: shear and dilation. For instance, for a 3-dimensional strain, infinitesimal affine deformations give the six independent strain quantities (e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{xz}, e_{yz}) (Love, 1927). For two dimensions, a complete basis set of affine transformations is shown in Fig. 2.

It is obvious that the equilibrium of a tensegrity structure holds for rigid-body motions, but it can also be straightforwardly shown for other affine transformations. Consider the equilibrium equation (1) written in the deformed coordinates $\bar{\mathbf{p}}$

$$\begin{aligned}\sum\nolimits_j \omega_{ij}\left(\bar{\mathbf{p}}_j - \bar{\mathbf{p}}_i\right) &= \sum\nolimits_j \omega_{ij}\left(\mathbf{U}\mathbf{p}_j + \mathbf{w} - \mathbf{U}\mathbf{p}_i - \mathbf{w}\right) \\ &= \sum\nolimits_j \omega_{ij}\left(\mathbf{U}\mathbf{p}_j - \mathbf{U}\mathbf{p}_i\right) \\ &= \mathbf{U}\sum_j \omega_{ij}\left(\mathbf{p}_j - \mathbf{p}_i\right) = \mathbf{0}\end{aligned} \tag{8}$$

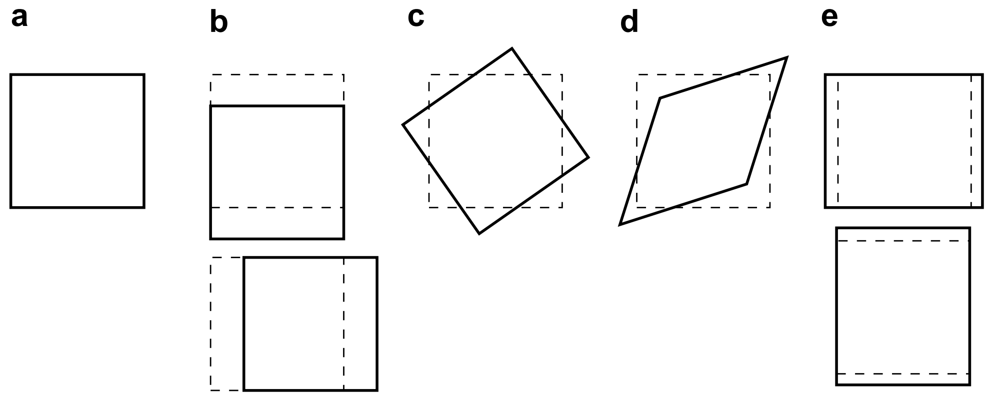


Fig. 2. The independent affine transformations of an object (a) in 2D space are: (b) two translations, (c) one rotation, (d) one shear, (e) two dilations. The total of 6 transformations complies with the $d(d+1)$ formula for $d=2$.

Thus, the affinely deformed configuration is in equilibrium. This knowledge can be used to great advantage in form finding to obtain new equilibrium shapes (Masic et al., 2005), but it also has important consequences for static balancing and the study of zero-stiffness modes. The above also provides an alternative view of the $\mathcal{N} \geqslant d(d+1)$ nullity requirement for $\tilde{\mathbf{\Omega}}$ found in form finding: there must be at least $d(d+1)$ affine transformations in the kernel of $\tilde{\mathbf{\Omega}}$ if a solution for the form finding is to be found in d-space.

3.2. Infinitesimal affine modes

So far we have only considered discrete affine transformations of coordinates, but we are really interested in continuous displacement paths and infinitesimal displacement vectors that are tangent to this path. When every configuration on the path is defined purely by an affine transformation of an original configuration, then we describe a tangent displacement vector to that path to be an *infinitesimal affine mode*.

Consider that there is path-length parameter λ, and the position of node i is given by $\mathbf{p}_i(\lambda)$, where the original configuration $\mathbf{p}_i = \mathbf{p}_i(0)$. If every configuration is an affine transformation of the original configuration, then we can write at each node i,

$$\mathbf{p}_i(\lambda) = \mathbf{U}(\lambda)\mathbf{p}_i + \mathbf{w}(\lambda) \tag{9}$$

where $\mathbf{U}(0) = \mathbf{I}$ and $\mathbf{w}(0) = \mathbf{0}$. To first order in λ, the parameters in Eq. (9) can be written as $\mathbf{U}(\lambda) = \mathbf{I} + \lambda\mathbf{V}$ and $\mathbf{w} = \lambda\mathbf{x}$. The infinitesimal affine mode of node i at the original configuration is then given by

$$\mathbf{d}_i = \left.\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}\lambda}\right|_{\lambda=0} = \mathbf{V}\mathbf{p}_i + \mathbf{x} \tag{10}$$

As for affine *transformations* of coordinates described in Section 3.1, the affine *mode* described by $\mathbf{V}$ and $\mathbf{x}$ can be split into infinitesimal rigid-body motions, and infinitesimal distortional components. Here, we split $\mathbf{V}$ into its symmetric component, $\mathbf{V}_Q$ and skew-symmetric component, $\mathbf{V}_S$.

$$\mathbf{V} = \mathbf{V}_Q + \mathbf{V}_S; \qquad \mathbf{V}_Q = \frac{\mathbf{V} + \mathbf{V}^{\mathrm{T}}}{2}, \quad \mathbf{V}_S = \frac{\mathbf{V} - \mathbf{V}^{\mathrm{T}}}{2} \tag{11}$$

The infinitesimal rigid-body motions are then described by $\mathbf{V}_S$ (rotations) and $\mathbf{x}$ (translations), and the distortional components are described by $\mathbf{V}_Q$.

Note that an infinitesimal affine mode is in the nullspace of the stress matrix for the structure, $\tilde{\mathbf{\Omega}}$. This can straightforwardly be proved along similar lines to Eq. (8), or simply by noting that because $\tilde{\mathbf{\Omega}}\mathbf{p}(\lambda) = \mathbf{0}$ for all λ, then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\tilde{\mathbf{\Omega}}\mathbf{p}(\lambda)) = \tilde{\mathbf{\Omega}}\frac{\mathrm{d}\mathbf{p}(\lambda)}{\mathrm{d}\lambda} = \mathbf{0} \tag{12}$$

and

$$\left.\frac{\mathrm{d}\mathbf{p}(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0} = \mathbf{d} \tag{13}$$

where $\mathbf{d}$ is an infinitesimal affine mode, which, from Eq. (12) therefore lies in the nullspace of $\tilde{\mathbf{\Omega}}$. If the stress matrix is of maximal rank, the nullspace is of dimension $d(d+1)$, and is hence fully described by the $d(d+1)$ independent infinitesimal affine modes defined by Eq. (10).

It should be noted that deforming an actual physical structure under an arbitrary affine transformation will generally speaking require work, will thus change the internal tensions, and will hence change the stress matrix. The condition under which an infinitesimal affine mode requires no work, and therefore has zero stiffness, is discussed next.

3.3. Statically balanced zero-stiffness modes

Recall that a structure has a zero stiffness if a given infinitesimal displacement vector $\mathbf{d}$ is in the nullspace of the tangent stiffness matrix $\mathbf{K}_t$, i.e.,

$$\mathbf{K}_t\mathbf{d} = \hat{\mathbf{K}}\mathbf{d} + \tilde{\mathbf{\Omega}}\mathbf{d} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{d} + \tilde{\mathbf{\Omega}}\mathbf{d} = \mathbf{0} \tag{14}$$

We focus here on the situation where both $\mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{d}$ and $\tilde{\mathbf{\Omega}}\mathbf{d}$ are zero. We shall exclude internal mechanisms by only considering tensegrity structures that when built with solely conventional elements would be stable for the given state of self-stress. Conventional elements are here understood to be tensile or compressive members that have a positive modified axial stiffness. Consequently, any zero-stiffness modes would be a result of the use of zero-free-length springs.

As shown in Section 3.2, infinitesimal affine modes lie in the nullspace of $\tilde{\mathbf{\Omega}}$. For a conventional structure these modes, that are not rigid-body motions, are stiffened by the modified material stiffness matrix $\hat{\mathbf{K}}$. For structures with zero-free-length springs, however, the positive *semi*-definiteness of $\hat{\mathbf{G}}$ and the resulting increased nullity in $\hat{\mathbf{K}}$ may result in new zero-stiffness modes. The key therefore lies in understanding the solutions to $\hat{\mathbf{K}}\mathbf{d} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{d} = \mathbf{0}$.

Consider an infinitesimal displacement $\mathbf{d}$ which is, to first order, length-preserving for the conventional members. Then $\mathbf{e} = \mathbf{A}^{\mathrm{T}}\mathbf{d}$ will have zero components for those conventional members, but may have non-zero components for the zero-free-length springs. If $\mathbf{e}$ is now premultiplied by $\hat{\mathbf{G}}$, $\hat{\mathbf{G}}\mathbf{e} = \hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{d}$, the resultant vector will also have zero components for the zero-free-length springs due to the zero modified axial stiffness on the diagonal of $\hat{\mathbf{G}}$. Thus, a displacement $\mathbf{d}$ that preserves the length of conventional elements will satisfy $\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{d} = \mathbf{0}$ and will hence be in the nullspace of $\mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}$.

From the above it now clearly follows that for an infinitesimal affine mode that preserves the length of conventional members, both $\hat{\mathbf{K}}\mathbf{d}$ and $\tilde{\mathbf{\Omega}}\mathbf{d}$ are zero and there exists a statically balanced zero-stiffness mode. This is illustrated by the simple statically balanced structure shown in Fig. 3.

We have thus far only considered infinitesimal displacements. This leads to the question of whether the statically balanced zero-stiffness modes are actually tangent to a *finite* zero-stiffness path. The next section will show that this is indeed the case.

3.4. Finiteness of statically balanced zero-stiffness modes

We will show that, once a structure is known to have a statically balanced zero-stiffness mode, then following any non-degenerate affine transformation of the coordinates of the structure, the new structure will also have a related statically balanced zero-stiffness mode.

Assume that we have an affine mode where for each node i,

$$\mathbf{d}_i = \mathbf{V}\mathbf{p}_i \tag{15}$$

where the mode preserves the length of conventional members, i.e., for every conventional member $\{i,j\}$, the infinitesimal relative displacement $\mathbf{d}_j - \mathbf{d}_i$ is orthogonal to the direction of the member, $\mathbf{p}_j - \mathbf{p}_i$,

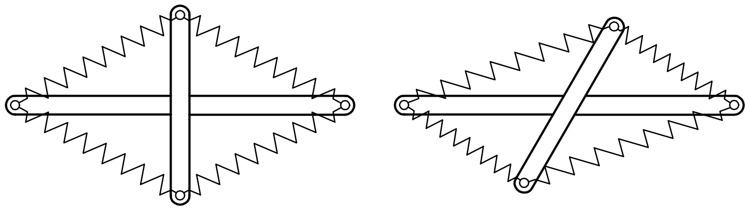


Fig. 3. Example of a 2D statically balanced structure consisting of two unconnected bars of differing lengths, and four zero-free-length springs of equal stiffness. When the bars are rotated with respect to each other, they remain in equilibrium and their movement thus has zero stiffness. In this example it is clear that the statically balanced mode is a combined shear and scale operation which preserves the bar lengths. Figure adapted from Herder (2001).

$$(\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}(\mathbf{d}_j - \mathbf{d}_i) = (\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}\mathbf{V}(\mathbf{p}_j - \mathbf{p}_i) = 0 \tag{16}$$

We know from Section 3.3 that this is a statically balanced zero-stiffness mode.

Now consider an affine transformation, so that the new structure has nodal coordinates $\bar{\mathbf{p}}_i$. For brevity, we will neglect rigid-body motions, which clearly preserve zero-stiffness, so that,

$$\bar{\mathbf{p}}_i = \mathbf{U}_Q\mathbf{p}_i \tag{17}$$

Where we assume that the transformation is non-degenerate, and so $\mathbf{U}_Q$ is invertible—we are not squashing the structure into a lower dimension. We shall show that $\bar{\mathbf{d}}_i = \mathbf{U}_Q^{-\mathrm{T}}\mathbf{d}_i$ (where $\mathbf{U}_Q^{-\mathrm{T}} = (\mathbf{U}_Q^{\mathrm{T}})^{-1} = (\mathbf{U}_Q^{-1})^{\mathrm{T}}$) is an affine mode of the transformed structure, that again preserves the lengths of conventional members.

It is straightforward to show that $\bar{\mathbf{d}}_i$ is an affine mode, as it can be written in terms of the new coordinates $\bar{\mathbf{p}}_i$ in the form given by Eq. (10),

$$\bar{\mathbf{d}}_i = \mathbf{U}_Q^{-\mathrm{T}}\mathbf{d}_i = (\mathbf{U}_Q^{-\mathrm{T}}\mathbf{V})\mathbf{p}_i = (\mathbf{U}_Q^{-\mathrm{T}}\mathbf{V}\mathbf{U}_Q^{-1})\bar{\mathbf{p}}_i \tag{18}$$

When considering the nodal coordinates of bar $\{i,j\}$ after the affine transformation, then

$$\bar{\mathbf{p}}_j - \bar{\mathbf{p}}_i = \mathbf{U}_Q(\mathbf{p}_j - \mathbf{p}_i) \tag{19}$$

and with the infinitesimal relative displacements in the new configuration given as

$$\bar{\mathbf{d}}_j - \bar{\mathbf{d}}_i = \mathbf{U}_Q^{-\mathrm{T}}(\mathbf{d}_j - \mathbf{d}_i) \tag{20}$$

then if we now consider the orthogonality equation (16) in the transformed configuration,

$$(\bar{\mathbf{p}}_j - \bar{\mathbf{p}}_i)^{\mathrm{T}}(\bar{\mathbf{d}}_j - \bar{\mathbf{d}}_i) = 0 \tag{21}$$

and then rewrite in terms of the original configuration

$$(\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}\mathbf{U}_Q^{\mathrm{T}}\mathbf{U}_Q^{-\mathrm{T}}\mathbf{V}(\mathbf{p}_j - \mathbf{p}_i) = 0 \tag{22}$$

we observe that $\mathbf{U}_Q^{\mathrm{T}}\mathbf{U}_Q^{-\mathrm{T}}$ cancels out, and thus the orthogonality also holds in the transformed configuration. This concludes our proof that if there exists an affine mode that preserves the length of conventional members in the original configuration, then there will again exist such an affine mode in any affinely transformed configuration. It clearly extends to proving the finiteness of the found statically balanced zero-stiffness mode, as in the affinely transformed configuration there will again exist such a mode, and thus the infinitesimal zero-stiffness modes connect to form a finite zero-stiffness path.

It is interesting to remark that throughout the finite affine displacement path of a zero stiffness tensegrity structure, the stress matrix, and therefore the tension coefficients of each of the members, will remain constant. For zero-free-length springs their tension coefficient is equal to their spring stiffness, and will therefore obviously remain constant. For conventional members, however, the only way their tension coefficient can remain constant is when their length remains unchanged. By this reasoning we again arrive at our previous requirement for the statically balanced zero-stiffness modes.

3.5. Are affine modes the only zero-stiffness modes?

This section describes the three conditions under which the infinitesimal affine modes that preserve the length of conventional members are the only possible zero-stiffness modes for the structure.

We firstly require that the structure is prestress stable when made from conventional members. This excludes the possibility of unstiffened internal mechanisms (where to first order the structure deforms without any members changing length). Secondly, we require that the stress matrix is of maximal rank, as described in Section 2.1. This ensures that the only vectors in the nullspace of the stress matrix are infinitesimal affine modes. Thirdly, we require that the stress matrix is positive semi-definite. This ensures that there are no negative eigenvalues in the stress matrix that can cause zero stiffness by the contributions of $\hat{\mathbf{K}}$ and $\tilde{\mathbf{\Omega}}$ cancelling each other out: $\hat{\mathbf{K}}\mathbf{d} = -\tilde{\mathbf{\Omega}}\mathbf{d}$. Any negative eigenvalues in the stress matrix are generally considered undesirable and should be avoided when designing tensegrity structures. In fact these requirements are not very restrictive: they describe 'superstable' tensegrities (Connelly, 1999), and these include most 'classic' tensegrity structures.

4. Length-preserving affine transformations

In the previous section it has been shown that an affine mode preserving the length of conventional members is a statically balanced zero-stiffness mode. In this section we will show that such a transformation exists if and only if the directions of the conventional members lie on a projective conic. This geometric interpretation provides several interesting observations regarding zero stiffness tensegrity structures.

4.1. Length-preserving affine transformation and projective conics

In order to understand under which circumstances the length of a member increases, decreases or stays the same under an affine transformation, we shall investigate the squares of the lengths of member $\{i,j\}$ under the affine transformation given by $\bar{\mathbf{p}}_i = \mathbf{U}\mathbf{p}_i + \mathbf{w}$, where $\mathbf{U}$ is an invertible d-by-d matrix, $\mathbf{w}$ a d-component column vector, and $\mathbf{p}_i$, $\mathbf{p}_j$ are the nodal coordinates:

$$\begin{aligned} L^2 - L_0^2 &= \left|(\mathbf{U}\mathbf{p}_j + \mathbf{w}) - (\mathbf{U}\mathbf{p}_i + \mathbf{w})\right|^2 - \left|\mathbf{p}_j - \mathbf{p}_i\right|^2 \\ &= (\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}\mathbf{U}(\mathbf{p}_j - \mathbf{p}_i) - (\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}\mathbf{I}^d(\mathbf{p}_j - \mathbf{p}_i) \\ &= (\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}[\mathbf{U}^{\mathrm{T}}\mathbf{U} - \mathbf{I}^d](\mathbf{p}_j - \mathbf{p}_i) \\ &= \mathbf{v}^{\mathrm{T}}\mathbf{Q}\mathbf{v} \end{aligned}$$

where $\mathbf{I}^d$ denotes the d-dimensional identity matrix, and $\mathbf{v} = (\mathbf{p}_j - \mathbf{p}_i)$ is the member direction. From this calculation it is clear that the symmetric matrix $\mathbf{Q} = \mathbf{U}^{\mathrm{T}}\mathbf{U} - \mathbf{I}^d$ and its associated quadratic form determine when member lengths increase, decrease or stay the same. We are interested in the situation where $\mathbf{v}^{\mathrm{T}}\mathbf{Q}\mathbf{v} = 0$.

It is obvious that a rigid-body motion will preserve the length of all members. This also follows clearly from the equation above, as an orthogonal matrix $\mathbf{U}_S$ will by definition give the identity matrix, $\mathbf{U}_S^{\mathrm{T}}\mathbf{U}_S = \mathbf{I}^d$, and will thus satisfy the equation, because $\mathbf{Q} = \mathbf{0}$. For the distortional affine transformations, we have to take a closer look at the quadratic form of $\mathbf{Q}$. For the case of $d = 3$, with directions $\mathbf{v}^{\mathrm{T}} = [v_x v_y v_z]$ and components of the symmetric $\mathbf{Q}$ given as $q_{kl} = q_{lk}$, this would take the following form

$$v_x^2 q_{11} + v_y^2 q_{22} + v_z^2 q_{33} + 2v_x v_y q_{12} + 2v_x v_z q_{13} + 2v_y v_z q_{23} = 0 \tag{23}$$

Eq. (23) describes the surface of a projective conic (Brannan et al., 1999). We shall rephrase by stating that a set of directions defined by

$$C = \{\mathbf{v} \in \mathbb{E}^d | \mathbf{v}^{\mathrm{T}}\mathbf{Q}\mathbf{v} = 0\} \tag{24}$$

forms a projective conic. This conic is clearly defined since scalar multiples of a vector satisfy the same quadratic equation, including the reversal of direction by a negative scalar. In degenerate cases $\mathbf{Q}$ could determine

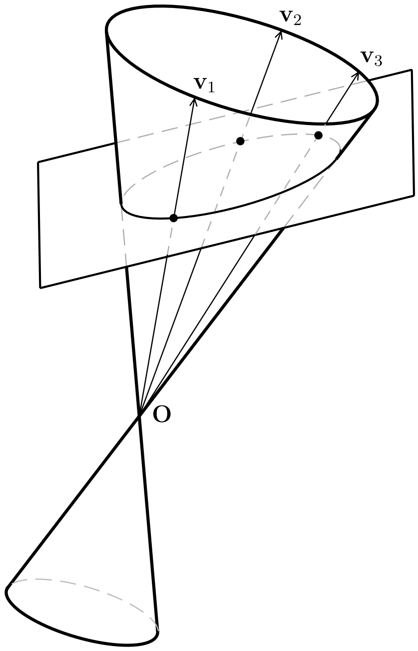


Fig. 4. The intersection of a plane with (one or two nappes of) a cone generates a conic section, which in non-degenerate cases is a quadratic curve such as an ellipse, parabola or hyperbola (Weisstein, 1999). The directions $\mathbf{v}_i$ on the conic project onto points on the conic section.

a single plane, or two planes through the origin. However, generally one would expect C to be the set of lines from the origin to the points of, for example, an ellipse in some plane not through the origin (see Fig. 4).

Supposing D is a set of directions in d-space, then there is an affine transformation $\bar{\mathbf{p}}_i = \mathbf{U}\mathbf{p}_i + \mathbf{w}$ that is not a rigid-body motion and that preserves lengths in the directions in D if and only if the directions in D lie on a projective conic. Or conversely, when the directions of certain members (in our case conventional elements) lie on a conic given by $\mathbf{Q} = \mathbf{U}^{\mathrm{T}}\mathbf{U} - \mathbf{I}^d$, their length will remain constant under the affine transformation $\mathbf{U}$. Analogously to Section 3.2, this can be extended to the infinitesimal affine modes $\mathbf{V}$. In fact, the conic form was already visible in the orthogonality Eq. (16).

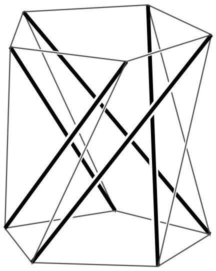
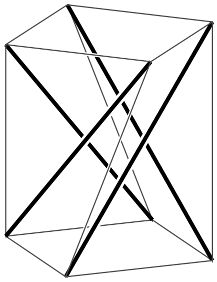
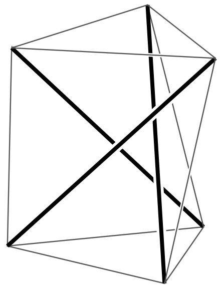
Of interest here are structures where all the conventional member directions lie on a projective conic, as the corresponding affine transformations will have zero stiffness. This is for instance clear for the structures shown in Table 1, where all the bar directions lie on a conic and the other members are zero-free-length springs. This leads to the observation that all the rotationally symmetric tensegrity structures discussed by Connelly and Terrell (1995) can have zero stiffness, when the cables are replaced by appropriate zero-free-length springs.

4.2. Number of zero-stiffness modes

Using the conic form, the number of independent length-preserving affine transformations, and thus zero-stiffness modes, of a structure can be determined. It is convenient to first consider a conic section, as shown in Fig. 4. It holds that five points in a plane—no three of which collinear—uniquely determine a conic (Brannan et al., 1999). This follows from the fact that a conic section is a quadratic curve. If fewer points are given, the conic is no longer uniquely defined and there exists more than one quadratic curve, and thus projective conic, that satisfies the points. In fact there are infinitely many solutions, but the number of independent conics is linked to the number of extra points needed for uniqueness.

To provide a more thorough analysis of the possible projective conics of a given structure, consider writing Eq. (23) simultaneously for each of the conventional members in a structure. If a structure has n conventional members, with member directions $\mathbf{v}_i^{\mathrm{T}} = [v_{xi} v_{yi} v_{zi}]$, this gives the following matrix equation:

Table 1
Number of statically balanced zero-stiffness modes for several rotationally symmetric tensegrity structures

			
Bar directions on conic	5	4	3
Zero-stiffness modes	1	2	3

All bar directions lie on a conic, and thus when replacing all cables by appropriate zero-free-length springs the structures will have zero stiffness. The number of bar directions on the conic and the number of zero-stiffness modes fit the counting rule established in Section 4.2.

$$\begin{bmatrix} v_{xi}^2 & v_{yi}^2 & v_{zi}^2 & 2v_{xi}v_{yi} & 2v_{xi}v_{zi} & 2v_{yi}v_{zi} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{xn}^2 & v_{yn}^2 & v_{zn}^2 & 2v_{xn}v_{yn} & 2v_{xn}v_{zn} & 2v_{yn}v_{zn} \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{22} \\ q_{33} \\ q_{12} \\ q_{13} \\ q_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{25}$$

It is now the dimension of the nullspace of the n-by-6 matrix defined in Eq. (25) that gives the number of independent conics (an arbitrary scaling of all of the constants q_{kl} in Eq. (23) does not define a different conic, and hence it is not the number of independent solutions to Eq. (25) that is important, but the dimension of the subspace of solutions).

If the rank of the matrix is r, the dimension of the nullspace is $6 - r$, and this will be the number of independent conics, and hence the number of zero-stiffness modes. For structures with more than 5 unique member directions, that all lie on a single projective conic, the rank will always be 5. For structures with 5 or less unique member directions, the rank is at most equal to the number of unique member directions. For many simple tensegrity structures, merely counting the number of unique member directions will suffice to determine the number of zero-stiffness modes, without performing any actual calculations.

4.3. Finite affine modes revisited

In geometry it is known that any conic (section) of specific type (parabola, hyperbola, ellipse) is affinely congruent to another conic of that type, and all non-degenerate conics are projectively congruent (Brannan et al., 1999). In practical terms this means that when affinely deformed, a conic will always remain a conic. This provides another approach to the finiteness of the length-preserving affine modes observed in Section 3.4. If all conventional member directions lie on a conic, there exists a length-preserving affine mode which has zero stiffness. If that mode is followed, the affinely deformed structure will again lie on a projective conic, and will again have a length-preserving affine mode with zero stiffness. Consequently, the zero-stiffness mode will be finite.

The proof is straightforward and follows a very similar route to the finiteness proof in Section 3.4. Supposing that a member direction $\mathbf{v}_i$ lies on the projective conic given by $\mathbf{v}^{\mathrm{T}}\mathbf{Q}\mathbf{v} = 0$, then considering an affine transformation

$$\bar{\mathbf{p}}_i = \mathbf{U}_Q\mathbf{p}_i$$

the member direction in the affinely deformed configuration, $\bar{\mathbf{v}}_i$ is then given by:

$$\bar{\mathbf{v}}_i = \mathbf{U}_Q\mathbf{v}_i \tag{26}$$

Writing the inverse as $\mathbf{v}_i = \mathbf{U}_Q^{-1}\bar{\mathbf{v}}_i$ and substituting into the original conic equation, we now obtain:

$$\bar{\mathbf{v}}_i^{\mathrm{T}}\mathbf{U}_Q^{-\mathrm{T}}\mathbf{Q}\mathbf{U}_Q^{-1}\bar{\mathbf{v}}_i = 0 \tag{27}$$

which again satisfies a conic equation in the deformed configuration, and hence concludes the proof. The new projective conic is described by the symmetric $\bar{\mathbf{Q}} = \mathbf{U}_Q^{-\mathrm{T}}\mathbf{Q}\mathbf{U}_Q^{-1}$.

5. Example

This section describes the numerical analysis of the classic tensegrity structure shown in Fig. 5. Both the nature and number of the calculated zero-stiffness modes fit the theory laid down in previous sections. This is further illustrated by the construction of a physical model.

5.1. Numerical analysis

It is expected that when the cables are replaced by zero-free-length springs, the structure will have three zero-stiffness modes, and that these modes are affine modes preserving the length of the three bars. This follows from the observation that the structure has three independent bar directions, and thus by Section 4.2 there are three independent zero-stiffness modes.

The tangent stiffness of the structure has been found using the formulation of Eq. (6) for two different cases. Firstly, with the structure consisting of conventional elements, and secondly, when made from conventional compressive bars, but using zero-free-length springs as tension members. The equilibrium configuration has been calculated with the analytical solution of Connelly and Terrell (1995), and the level of self-stress—and thus the stress matrix—is identical for both cases. All conventional elements have a ‘stiffness’ of $EA = 100$ N, the horizontal springs 1 N/m and the vertical springs $\sqrt{3}$ N/m. The internal tension of the structure is uniquely prescribed by these spring stiffnesses; the stiffness ratio is a property of this structure and is independent of the structure’s radius or height. The results are presented as the stiffness of each of the eigenmodes (excluding rigid-body motions) of $\mathbf{K}_t$ in Table 2a and b.

For the conventional structure all eigenvalues of the tangent stiffness matrix are positive, and the stress matrix is of maximal rank. The system has an internal mechanism, which is stabilized by the state of self-stress. This can be seen in the first line of Table 2a, where the $\hat{\mathbf{K}}$ component is almost zero (it is not precisely zero because the eigenvectors of $\hat{\mathbf{K}}$ and $\mathbf{K}_t$ are not precisely aligned).

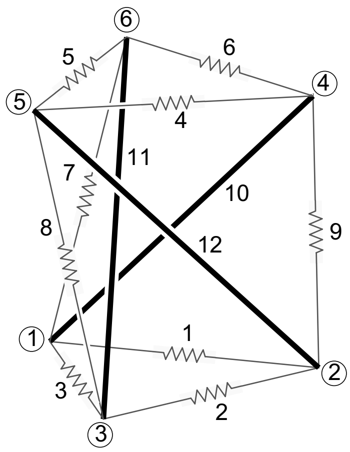


Fig. 5. Rotationally symmetric tensegrity structure. The structure has a circumscribing radius $R = 1$ m, height $H = 2$ m and the two parallel equilateral triangles (nodes 1–3 and nodes 4–6) are rotated $\pi/6$ with respect to each other.

Table 2
Stiffness of each of the eigenmodes of $\mathbf{K}_t$, excluding rigid-body motions, for (a) the conventional structure and (b) the structure with zero-free-length springs as tension members

$\mathbf{K}_t$ (N/m)	$\hat{\mathbf{K}}$ (N/m)	$\tilde{\mathbf{\Omega}}$ (N/m)
(a)		
5.6304	0.0174	5.6130
27.8384	26.1960	1.6424
27.8384	26.1960	1.6424
83.2190	79.1954	4.0236
83.2190	79.1954	4.0236
107.3763	103.0749	4.3014
107.3763	103.0749	4.3014
113.8525	113.5350	0.3175
132.5068	130.4743	2.0325
132.5068	130.4743	2.0325
176.2051	170.2051	6.0000
225.4577	225.3881	0.0696
(b)		
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
5.6703	0.0267	5.6436
5.6703	0.0267	5.6436
5.7899	0.0174	5.7724
6.0000	0.0000	6.0000
6.0000	0.0000	6.0000
6.0000	0.0000	6.0000
75.5997	75.3721	0.2276
75.7193	75.3629	0.3564
75.7193	75.3629	0.3564

The total stiffness $\mathbf{K}_t$ is the sum of the contributions of $\hat{\mathbf{K}}$ and $\tilde{\mathbf{\Omega}}$.

When zero-free-length springs are placed in the structure, three new zero-stiffness modes appear in $\mathbf{K}_t$—the first three rows of Table 2b—which are linearly dependent on the affine transformations for shear and dilation. These modes can be considered in a symmetry-adapted form (Kangwai and Guest, 1999) as a totally symmetric mode, and a pair of modes that are symmetric and antisymmetric with respect to a dihedral rotation. The fully symmetric mode is shown in Fig. 6. It is purely dependent on scaling transformations, and corresponds to a mode where the structure is compressed in the x–y plane and expands in the z-direction.

In conclusion, the numerical results confirm the theoretical predictions: the zero-stiffness modes correspond to affine transformations, the bar lengths remain constant—$\mathbf{A}^{\mathrm{T}}\mathbf{d}$ returned zero for the bars—and the number of introduced zero-stiffness modes fits the counting rule from Section 4.2.

Now imagine taking the structure in Fig. 5 and replacing some of the zero-free-length springs by cables. Then each replaced spring will reduce the number of zero-stiffness modes by one, up to the point where (after three added cables) the conventional elements no longer all lie on a projective conic, wherefore the structure loses its zero stiffness and becomes rigid. This has implications for the possible application of this type of tensegrity as a type of parallel platform, where both top and bottom triangle are of fixed lengths (Baker and Crane, 2006). In that situation there will exist no displacement that has zero stiffness, and it will therefore not be possible to alter position and/or orientation at a constant potential energy level.

5.2. Physical model

To illustrate that the zero stiffness tensegrity structure is not merely mathematical, a demonstration prototype was constructed. It does not make use of actual zero-free-length springs, but of conventional springs that are attached alongside the bars such that the properties of zero-free-length springs are achieved. As gravity

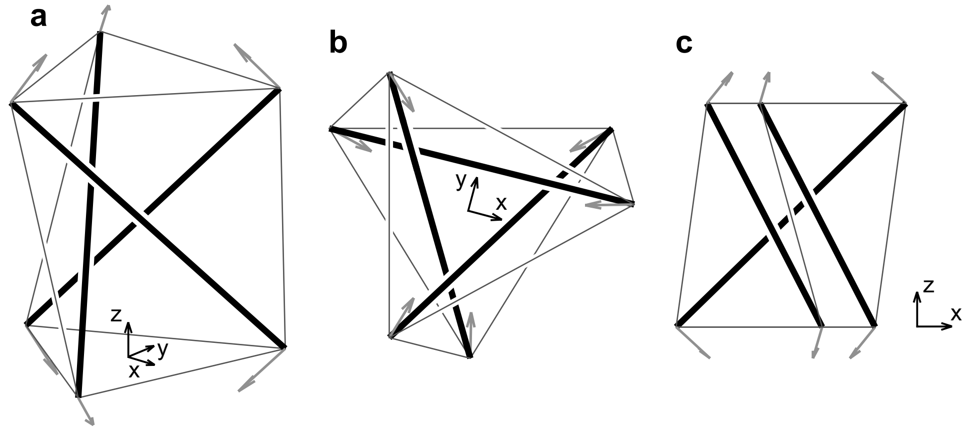


Fig. 6. Fully symmetric zero-stiffness mode, with (a) 3D view, (b) top view and (c) side view. All displacement vectors are of equal magnitude, and with equal z-component. In this mode the rotation angle between bottom and top triangle remains constant throughout the displacement.

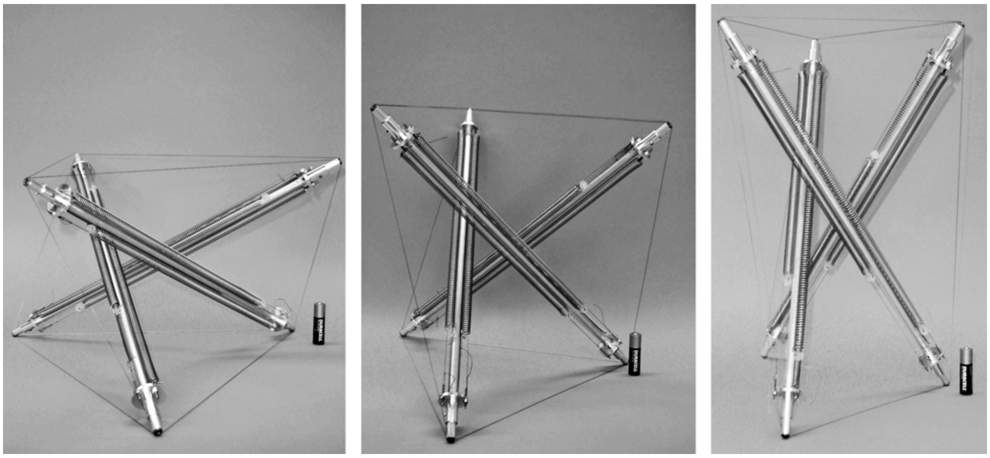


Fig. 7. The demonstration model deformed in accordance with the symmetrical zero-stiffness mode. Note that further and other (asymmetric) deformation is still possible.

forces were not taken into account in the calculations, if perfectly constructed, the prototype should collapse under its own weight. The friction in the system prevents this from happening, however. As a result the structure requires some external work to deform, but it will nevertheless remain in equilibrium over a wide range of positions (see Fig. 7). Further details of the construction of this model are described in Schenk et al. (2006).

6. Summary and conclusions

This paper has investigated the zero-stiffness modes introduced to tensegrity structures by the presence of zero-free-length springs. It was shown that in the absence of external loads and constraints, affine modes that preserve the length of conventional members are statically balanced zero-stiffness modes. Those modes involve changing spring lengths, but require no energy to move, even over finite displacements. For prestress stable tensegrities with a positive semi-definite stress matrix of maximal rank, we further showed that these are the only possible zero-stiffness modes introduced by the zero-free-length springs.

It was further shown that such length-preserving affine transformations are present if and only if the directions of the conventional elements lie on a projective conic. This geometric interpretation revealed an entire family of tensegrity structures that can exhibit zero stiffness, and led to a simple method for determining the number of independent length-preserving affine modes.

By only considering tensegrity structures, the theory in this paper has several inherent restrictions. Future work will attempt to resolve these aspects, starting with the inclusion of external loads and nodal constraints in the analysis of pin-jointed structures. The next phase would be to apply the acquired knowledge to non-pin-jointed structures, in order to describe statically balanced structures such as the 'Anglepoise' lamp in a generic way.

Finally, the construction of the physical model has illustrated that this type of structure is not yet suited for practical applications. Once difficulties such as accuracy of spring stiffness ratio, presence of friction and overall complexity of design have been overcome, a totally new class of structures, or mechanisms, will be available to engineers.

Acknowledgements

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